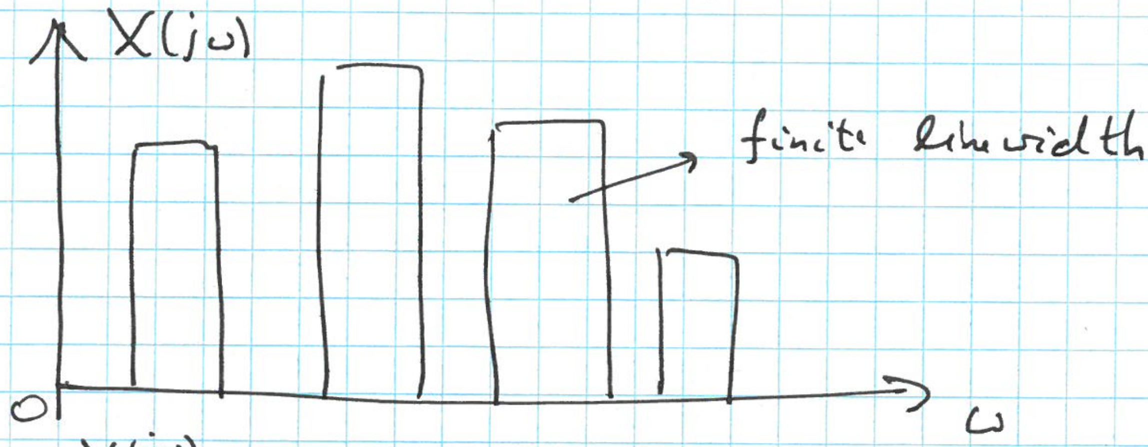
1. Fourier Transforms

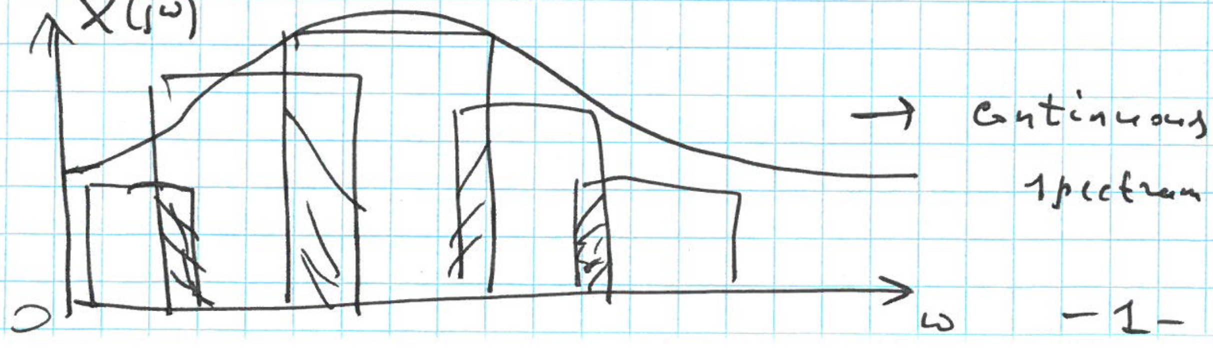
Periodic function can always be expanded into a finite number of harmonics $e^{\pm jn\omega}$

Example:

$$\cos\omega t = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

Nature example of an FT: rainbow





$$\sum_n X_{n\omega} e^{jn\omega t} \to \int_{-\infty}^{\infty} dX(\omega) e^{j\omega t} \to$$

$$\to \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \underbrace{X(j\omega)}_{\text{spectral density of signal } x(t)} e^{j\omega t}$$

$\therefore$ FT

$$\boxed{\begin{aligned} x(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} X(j\omega) e^{j\omega t} \quad (\text{inverse FT}) \\ X(j\omega) &= \int_{-\infty}^{\infty} dt\, x(t) e^{-j\omega t} \quad (\text{direct FT}) \end{aligned}}$$

Properties:

1) Shift $\quad x(t-t_0) \overset{FT}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$

Consider $\quad X(j\omega) = \int_{-\infty}^{\infty} dt\, x(t) e^{-j\omega t}$

Take $\quad \int_{-\infty}^{\infty} dt\, x(t-t_0) e^{-j\omega t} \quad \left| \tau = t - t_0 \right| =$

$$= \underbrace{\int_{-\infty}^{\infty} d\tau\, x(\tau) e^{-j\omega\tau}}_{X(j\omega)} e^{-j\omega t_0} = e^{-j\omega t_0} X(j\omega)$$

2) Complex conjugation

$$\mathcal{F}[x^*(t)] = X^*(-j\omega)$$

$$x^*(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} X^*(j\omega)\, e^{-j\omega t} \Big|_{\omega \to -\omega} =$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} X^*(-j\omega)\, e^{j\omega t}$$

$$\therefore \quad \mathcal{F}[x^*(t)] = X^*(-j\omega)$$

3) Derivative

$$x'(t) \equiv \frac{dx}{dt} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} X(j\omega) \frac{d}{dt} e^{j\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} j\omega X(j\omega) \times$$

$$\times e^{j\omega t}$$

$$\therefore \quad \mathcal{F}[x'(t)] = j\omega X(j\omega)$$

4) Parseval theorem.

Introduce signal energy

$$E = \int_{-\infty}^{\infty} dt\, |x(t)|^2$$

$$E = \int_{-\infty}^{\infty} dt\, |x(t)|^2 = \int_{-\infty}^{\infty} dt\, x^*(t)\, x(t) = \int_{-\infty}^{\infty} dt\, x^*(t) \times$$

$$\times \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} X(j\omega)\, e^{j\omega t}$$

$$E = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} X(j\omega) \underbrace{\int_{-\infty}^{\infty} dt\, x^*(t)\, e^{j\omega t}}_{X^*(j\omega)} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |X(j\omega)|^2$$

The same energy can be expressed either in time domain or in the spectral domain! :

$$\boxed{\int_{-\infty}^{\infty} dt\, |x(t)|^2 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |X(j\omega)|^2}$$

5) Convolution theorem

$$y(t) = x(t) * h(t) \equiv \underbrace{\int_{-\infty}^{\infty} d\tau\, x(\tau)\, h(t-\tau)}_{\text{convolution of } x \,\&\, h}$$

Define

$$Y(j\omega) = \int_{-\infty}^{\infty} dt\, y(t)\, e^{-j\omega t}$$

$$Y(j\omega) = \int_{-\infty}^{\infty} dt\, e^{-j\omega t} \int_{-\infty}^{\infty} d\tau\, x(\tau)\, h(t-\tau) = \int_{-\infty}^{\infty} d\tau\, x(\tau)\, \times$$

$$\times \underbrace{\int_{-\infty}^{\infty} dt\, e^{-j\omega t} h(t-\tau)}_{H(j\omega) e^{-j\omega\tau}} = H(j\omega) \underbrace{\int_{-\infty}^{\infty} d\tau\, x(\tau)\, e^{-j\omega\tau}}_{X(j\omega)}$$

$\therefore$

$$\boxed{\mathcal{F}[x * h] = X(j\omega) H(j\omega)}$$

Convolutions describe response of linear, time-invariant systems (LTI) to external signal inputs

Examples: 1. Capacitor

$$q = Cv, \quad q = \int_0^t dt'\, i(t')$$

$$v = \frac{1}{C} q = \frac{1}{C} \int_0^t dt'\, i(t')$$

$$v = \int_{-\infty}^{\infty} d\tau\, i(\tau)\, h(t-\tau)$$

$$\boxed{h(t) = \frac{1}{C} u(t)}, \quad u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

2. Inductor

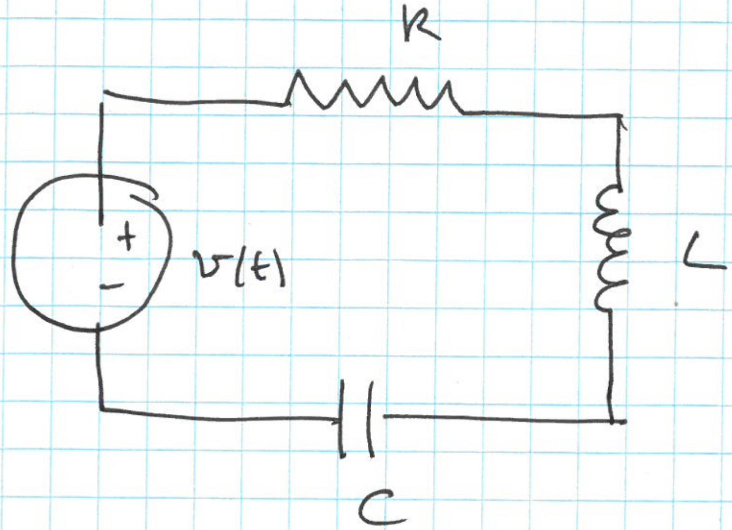
$$v = L \frac{di}{dt}$$

$$v = L \int_{-\infty}^{\infty} d\tau\, i(\tau) \left[-\frac{d}{dt} \delta(t-\tau) \right]$$

$\therefore$

$$\boxed{h(t) = -L \frac{d}{dt} \delta(t)}$$

LRC - circuit response with FT



$$\frac{q}{C} + R\dot{q} + L\ddot{q} = v(t) \quad , \quad \text{Kirchhoff}$$

FT:

$$Q(j\omega) = \int_{-\infty}^{\infty} dt\, q(t)\, e^{-j\omega t}$$

$$V(j\omega) = \int_{-\infty}^{\infty} dt\, v(t)\, e^{-j\omega t}$$

$$\frac{Q}{C} + j\omega R Q - \omega^2 Q L = V$$

$$\left(-\omega^2 + 1/LC + j\frac{R}{L}\omega\right) Q = V/L$$

Introduce $\quad 2\gamma = R/L, \quad \omega_0^2 = 1/LC$

$\therefore$

$$\left(-\omega^2 + \omega_0^2 + 2j\gamma\omega\right) Q = V/L$$

$$Q(j\omega) = H(j\omega)\, V(j\omega)$$

$\therefore$

$$H(j\omega) = \frac{1}{L(\omega_0^2 - \omega^2 + 2j\gamma\omega)}$$

Consider a good quality LRC: $\gamma << \omega_0$

Take ω to be close to ω_0 : $\omega \approx \omega_0$

$\therefore$ $\omega \approx \omega_0$, $\gamma << \omega_0$

$\therefore$

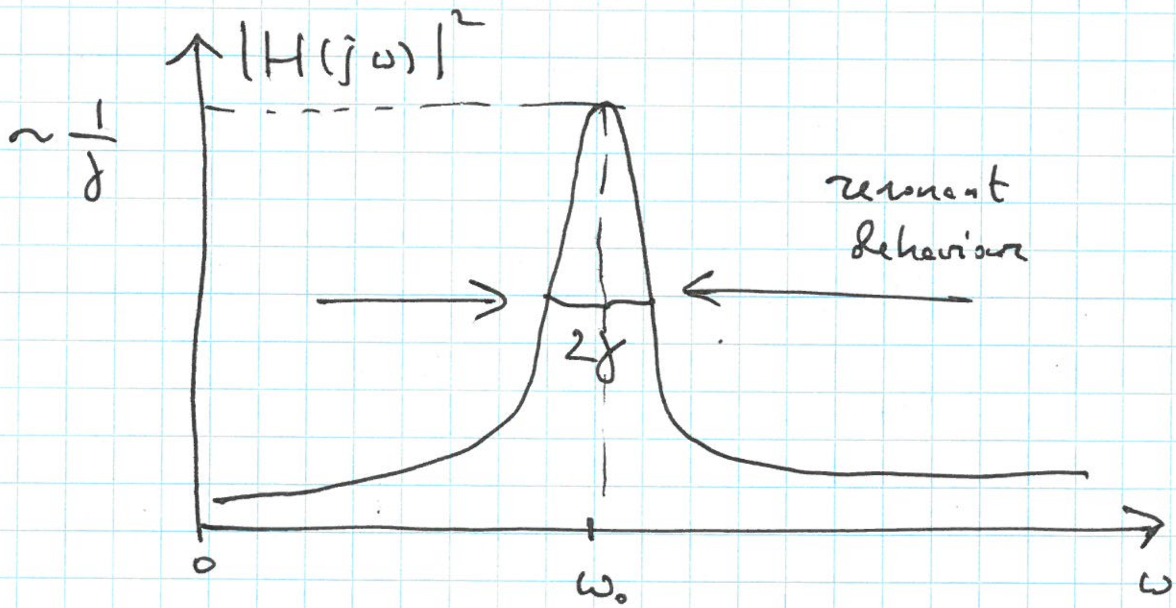
$$\omega_0^2 - \omega^2 + 2j\gamma\omega = (\omega_0 - \omega)(\omega_0 + \omega) + 2j\gamma\omega \approx$$
$$\approx 2\omega_0(\omega_0 - \omega) + 2j\gamma\omega_0 \approx 2\omega_0(\omega_0 - \omega + j\gamma)$$

$\therefore$

$$H(j\omega) \approx \frac{1}{2L\omega_0(\omega_0 - \omega + j\gamma)}$$

$\therefore$

$$|H(j\omega)|^2 \approx \frac{1}{4L^2\omega_0^2\left[(\omega_0 - \omega)^2 + \gamma^2\right]}$$



Laplace transform

Define:

$$\boxed{X(s) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-st} \qquad s = \sigma + j\omega}$$

Inverse:

$$X(s) = \int_{-\infty}^{\infty} dt\, \underbrace{x(t)\, e^{-\sigma t}}_{\tilde{x}(t)}\, e^{-j\omega t}$$

$$\tilde{x}(t) \equiv x(t)\, e^{-\sigma t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, X(\sigma + j\omega)\, e^{j\omega t}$$

$\therefore$

$$x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, X(\sigma + j\omega)\, e^{(\sigma + j\omega)t}$$

$\therefore$

$$\boxed{x(t) = \int_{\sigma - j\infty}^{\sigma + j\omega} \frac{ds}{2\pi j}\, X(s)\, e^{st}}$$

or

$$\boxed{x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} ds\, X(s)\, e^{st}}$$

Limiting properties :

1) shift : $\boxed{\mathcal{L}[x(t-t_0)] = e^{-st_0} X(s)}$

2) conjugation $\boxed{\mathcal{L}[x^*(t)] = X^*(s^*)}$

3) convolution $\boxed{\mathcal{L}[x_1 * x_2] = X_1(s) X_2(s)}$

4) derivative $\boxed{\mathcal{L}[\frac{d}{dt} x] = s X(s)}$

5) Finite value theorem

Suppose the signal $x(t)$ is switched on at $t=0$:

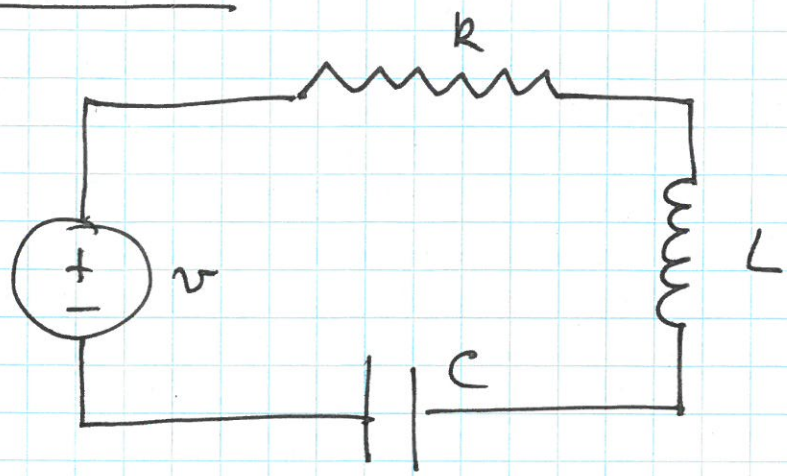
$$x(t) = \begin{cases} x(t), & t > 0 \\ 0, & t \le 0 \end{cases}$$

and <u>there exists a finite limit</u>

$$x_\infty = \lim_{t \to \infty} x(t) < \infty ,$$

$$\boxed{x_\infty = \lim_{t \to \infty} x(t) = \lim_{s \to 0} s X(s)}$$

Example 1 RLC-circuit revisited



$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = v$$

$$Q(s) = \int_{-\infty}^{\infty} dt\, q(t)\, e^{-st}$$

$$V(s) = \int_{-\infty}^{\infty} dt\, v(t)\, e^{-st}$$

$$\left(s^2 L + Rs + 1/C\right) Q = V$$

$\therefore$

$$Q = \frac{V/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\omega_0^2 = 1/LC, \quad 2\gamma = R/L$$

$\therefore$

$$Q = \frac{V/L}{s^2 + 2\gamma s + \omega_0^2}$$

$$Q = \frac{V/L}{(s+\gamma)^2 + \omega_0^2 - \gamma^2}$$

$$Q(s) = H(s)\,V(s)$$

$\therefore$

$$H(s) = \frac{1/L}{(s+\gamma)^2 + \omega_0^2 - \gamma^2}$$

Use tables:

$$\mathcal{L}^{-1}\left[\frac{\nu}{(s+\alpha)^2+\nu^2}\right] = u(t)\,e^{-\alpha t}\sin\nu t$$

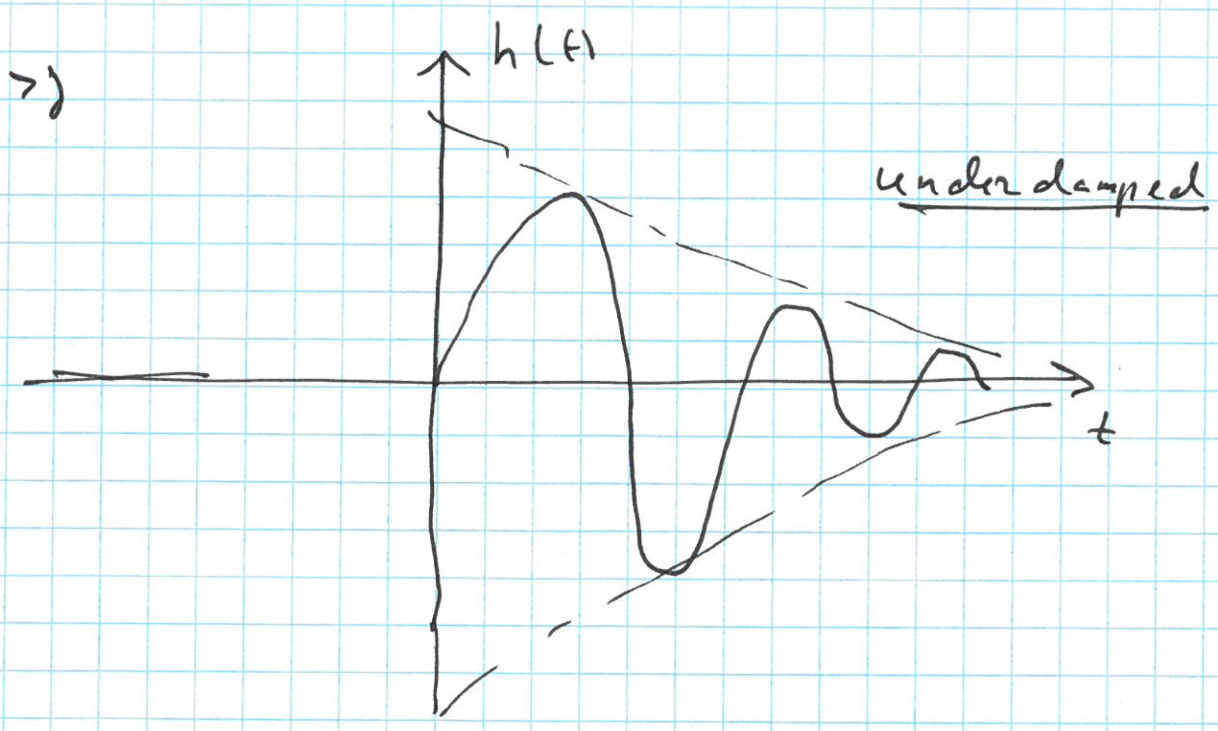
$\therefore$

$$h(t) = L^{-1}\,u(t)\,e^{-\gamma t}\,\frac{\sin\left(\sqrt{\omega_0^2-\gamma^2}\,t\right)}{\sqrt{\omega_0^2-\gamma^2}}$$

$\therefore$

$$h(t) = u(t)\,e^{-\gamma t}\,\frac{\sin\left(t\sqrt{\omega_0^2-\gamma^2}\right)}{L\sqrt{\omega_0^2-\gamma^2}}$$

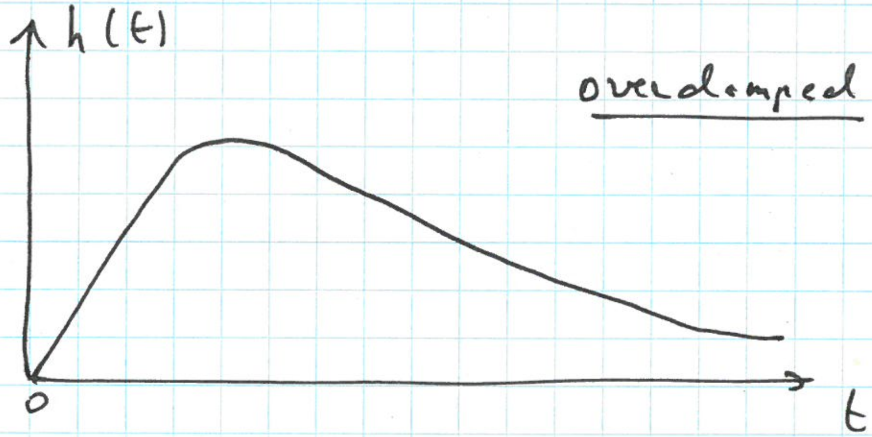
a) $\omega_0 > \gamma$



b) $\omega_0 < \gamma$ $\quad \sin jx \to j \sinh x$

$$\sqrt{\omega_0^2-\gamma^2} = j\sqrt{\gamma^2-\omega_0^2}$$

$$h(t) \to u(t)\, e^{-\gamma t} \frac{\sinh\left(t\sqrt{\gamma^2-\omega_0^2}\right)}{L\sqrt{\gamma^2-\omega_0^2}}$$



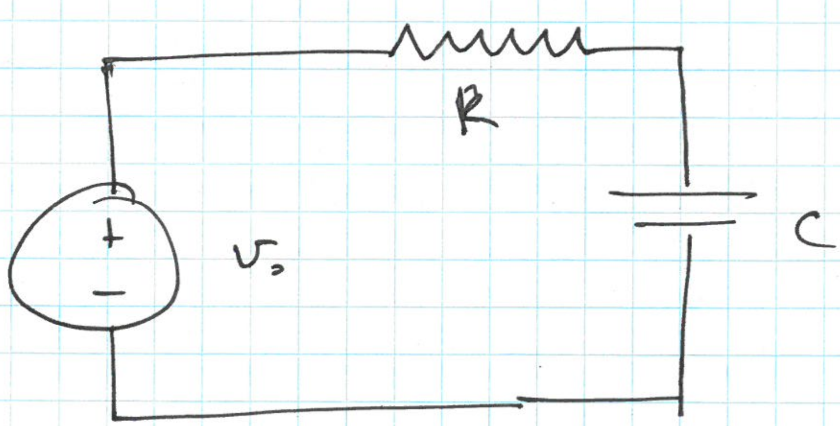
overdamped

c) $\omega_0 = \gamma$ $\quad h(t) = u(t)\, e^{-\gamma t} \lim\limits_{\omega_0 \to \gamma} \frac{\sin\left(t\sqrt{\omega_0^2-\gamma^2}\right)}{L\sqrt{\omega_0^2-\gamma^2}}$

$$\propto u(t)\, t\, e^{-\gamma t}$$

critically damped regime

Example 2 $\quad$ RC-circuit asymptotics



$$R\dot{q} + \frac{1}{C} q = v_0$$

$$\dot{q} + \frac{1}{\tau} q = v_0/R \quad , \quad \tau = RC$$

$$sQ + \frac{1}{\tau} Q = \frac{v_0}{sR}$$

$$\therefore \quad Q(s) = \frac{v_0/R}{s(s + 1/\tau)}$$

$$Q_\infty = \lim_{s \to 0} sQ(s) = \frac{v_0/R}{1/\tau} = C v_0$$

The capacitor will charge to the source ~~voltage~~.
voltage!