Math bootcamp: Combinatorics and probability

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I. COMBINATIONS AND PERMUTATIONS: WHAT'S THE DIFFERENCE?

Examples:

1. Combination of items in your lunch Chili combo (chili, bun, doughnut, tea/coffee), no particular order.

2. My cell phone number is an **ordered** combination of numbers.

- Whenever the order is **irrelevant**, it is a **<u>Combination</u>**

- Whenever the order **does** matter it is a **<u>Permutation</u>** $\Longrightarrow$

$$\boxed{\textbf{Permutation} \Longleftrightarrow \textbf{ordered combination!}}$$

A. Permutations

- *Permutations with repetitions*: number/letter passwords, lock combos;

- *Permutations with no repetitions*: student paper competition winners list (there's only one person in the first place etc.)

Permutations with repetitions: Say we've got n objects to choose from, that is n options. There're $n \times n$ ways to choose two objects out of n because there are exactly n options for each choice–repetitions are allowed–implying that there are exactly $\underbrace{n \times n \times \ldots \times n}_{r\text{ times}}$, i. e.,

$$\boxed{n^r \text{ways to choose } r \text{ out of } n} \tag{1}$$

Example: How many ways are there to pick 3–digit lock combinations?

<u>Solution</u>–*There are* $n = 10$ *digits to choose from, (0,1,2,3,4,5,6,7,8,9), and we need to pick any three, i.e.,* $r = 3$*. The order matters, but we may repeat digits. Hence the sought number is* 10^3.

Permutations with no repetitions: Suppose we have n objects to choose from and we seek the number of ways we can choose r objects each entering a combination only once. Clearly,

$$\boxed{\text{Lack of repetitions reduces the number of available options}}$$

For example, we've got 5 numbered billiard balls and we want to find out the number of ways we can arrange 3 of them on the table. Note the balls are different: they are numbered and the order matters! There are exactly 5 ways to choose the first ball. However, once the first has been picked, there are only 4 ways to pick the second as no repetitions are allowed and so on. Altogether, there are $5 \times 4 \times 3 = 60$ ways. Generalizing and introducing the notation $P(n, r)$ the number of permutations of r objects out of n,

$$\boxed{P(n,r) = n \times (n-1) \ldots \times (n-r+1) = \frac{n!}{(n-r)!}} \tag{2}$$

Example: How many arrangements of 3 **different** *digits are there, chosen from the ten digits 0 to 9 inclusive?*

<u>Solution</u>–Clearly it is a permutation because all digits are ordered (numbered) and no same digits are allowed so there are no repetitions; $n = 10$ *and* $r = 3$*, implying that* $P(10, 3) = 10!/(10-3)! = 8 \times 9 \times 10 = 720$.

Corollary: If $r = n$ the problem reduces to that of ordering n different objects. The answer is

$$\boxed{P(n,n) = n!} \tag{3}$$

B. Combinations

We will only consider **combinations with no repetitions**. A very popular example is furnished by lotteries: numbers are drawn one at a time and anyone having a lucky combination wins, regardless of the number order! We can deduce the answer from the number of permutations without repetitions. To this end we adopt the algorithm

- assume that the order does matter to get a number of permutations;
- alter it so the order does not matter.

Let's first illustrate the approach with an example. Let us revise our pool ball example by assuming the balls are not numbered. We seek the number of combination of 3 balls out of 6. Clearly, this number must be greatly reduced compared to $P(6, 3)$ because we couldn't care less about ordering. For instance, all ordered selections of balls #1, 2 and 3, correspond to just one (1,2,3) now, i.e.,

$$\underbrace{\{(1,2,3); (2,3,1); (1,3,2); (2,1,3); (3,2,1); (3,1,2)\}}_{=6}, \qquad \text{ordered combo,}$$

are equivalent to just one

$$(1, 2, 3), \qquad \text{no order}$$

There are exactly 3!=6 orderings of these three balls. Hence the number of combinations must be reduced by 3! when the order ceases to matter. Generalizing, the number of combinations of r objects out of n is given by the number of permutations divided by r!, that is

$$C_r^n = \begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!(n-r)!} \tag{4}$$

Example: solve the pool case.

Solution–Using the combo formula, $C_r^n = 6!/3!^2 = 4 \times 5 \times 6/6 = 20.$

II. PRACTICE COMBINATORICS PROBLEMS

Problem 1. Fun with student groups: *How many different student groups of 5 people can be chosen out of 10 students?*

Solution–The word "different" means we cannot have a group with the same student entering twice or more times, (unless said student is cloned) i.e., no repetitions are allowed. Otherwise, the student order is irrelevant here. We are after combinations with no repetitions, C_r^n with $n = 10$ and $r = 5$. Thus,

$$C_5^{10} = 10!/5!^2 = 252.$$

Problem 2. Committees with a Chair: *Linda is a committee Chair. In how many ways can a committee of 5 be chosen from 10 people given that Linda must chair it?*

Solution–Conceptually this problem is similar with the caveat that the options are limited by stipulating that Linda must be on each committee. In other words, we still seek C_r^n except $r = 4$ and $n = 9$ such that

$$C_4^9 = \frac{9!}{4!5!} = 126.$$

Problem 3. Pesky passwords: *A password consists of any two letters of the alphabet, followed by any three digits chosen from 0 to 9. How many different possible passwords are there?*

Solution–Letters and digits are distinct and the order is crucial to a password. Also, the word "any" implies we can use a letter or a digit as many times as we want, that is repeats are allowed. As well, letter and number combinations are independent so that we can work them out independently and multiply the results (think of a matrix with number combos entering as columns and letter combos as rows and we look for the total number of its elements). The total number permutations amount to $P(n, r) = n^r$ with $n = 10$ and $r = 3$ and the total letter permutations add up to $P(m, k) = m^k$ with $m = 26$ and $k = 2$, hence

$$P_{pswd} = 10^3 \times 26^2 = 676,000.$$

Problem 4. Word games: *Assuming that any arrangement of letters forms a 'word', how many 'words' of any length can be formed from the letters of the word MATH?*
Solution–All letters are distinct and different letter arrangements constitute different words. Hence we are dealing with permutations with no repetitions. We can form one-,two-, three- or four-letter words. There are exactly 4 ways to form one-letter words by picking just one letter. When we form a two-letter word we have 4 choices for the first letter and only 3 for the second, so the number of ways=4×3=12. By the same token for three-letter words, the number of ways=4×3×2=24. Finally, there are only 4! =24 four-letter words (permutations of 4 numbers). Overall, the number of permutations=4+12+24+24=64.

Note the following "rules of thumb" when use conjunctions "and" in problem 4 (combos of letters **and** numbers) and "or" in problem 5 (combos of either one- **or** two-letter words):

- combos/probabilities of one **and** other kinds $\Longleftrightarrow$ **multiplying** combos/probabilities;
- combos/probabilities of one **or** different kind $\Longleftrightarrow$ **adding** combos/probabilities.

Problem 5. Happy hour pizza: *Lucky Joe enters a pizza bar during a happy hour. The happy hour deal allows Joe to choose anyone of 4 different cheese, 6 meat and 5 veggie toppings on his pizza. He may have his pizza with either one, two, or three kinds of toppings. How many different pizza options are available to Joe if he wants the happy hour deal?*
Solution–If Joe chooses only one topping kind (cheese, meat or veggie), he has 4+6+5=15 options available. If he chooses to have a pizza with two different kinds of toppings, he has 5×4 or 5×6 or 4×6=5× 4+ 5×6+ 4×6=74 options. Finally, if he chooses all three toppings, he's got 4 and 5 and 6 combos=4×5×6=120 options. Overall, he has 15+74+120= 209 options. Lucky dude!

III. PROBABILITY BASICS

We define the probability of an event A as a limit of the ratio of the number of successes S leading to event A to the total number N of trials:

$$\boxed{P_N = \frac{S}{N} = \frac{\#\text{successes}}{\#\text{trials}}}.$$

If there exists a limit as $N \to \infty$, then the probability of success P is defined as

$$P = \lim_{N\to\infty} P_N.$$

In practice, we will assume that the number of trials is sufficiently large to ensure the existence of the limit, and use $P = S/N$ as the probability of success **definition**.

Examples:

Example 1. Urn and balls: *An urn contains a white and b black balls. Someone randomly draws a ball from the urn, checks out its colour and leaves it aside. The ball happens to be white. What is the probability to draw another white ball from the same urn?*

Solution–An event here is drawing a ball and a favourable outcome of the event in our case is drawing a white ball. As one white ball has already been picked up, the number of favourable outcomes shrinks to $a - 1$; the total number of outcomes is $(a + b - 1)$ (overall number of balls). Hence by definition, $P = (a - 1)/(a + b - 1)$.

Example 2. Urn and ball combos: *An urn contains $a \geq 2$ white and b black balls. Someone draws two balls from the urn at random. What is the probability that both balls are white?*

Solution–As the balls are the same and the order is irrelevant, the number of favourable outcomes (2 white balls) corresponds to a number of combinations of two out of a whites, i. e., $S = C_2^a$ whereas the total number of outcomes is a combo number of two out of all balls in the urn: $N = C_2^{a+b}$. Hence,

$$P = \frac{C_2^a}{C_2^{a+b}} = \frac{(a+b-2)!}{(a+b)!}\,\frac{a!}{(a-2)!} = \frac{a(a-1)}{(a+b-1)(a+b)}.$$

Example 3. Quality control: *There are l defective items in a product party of k. A quality inspector picks r items at random. What is the probability that s of them are defective?*

Solution–If there are s defective items in the selected party of r, there must be $r - s$ good items. The problem then reduces to finding the number of combinations of choosing s items out of l and $r-s$ items out of $k-l$. Since the order is immaterial the answer is $C_s^l C_{r-s}^{k-l}$; thus, $S = C_s^l C_{r-s}^{k-l}$. On the other hand, the total number of outcomes is $N = C_r^k$. It then follows,

$$P = S/N = C_s^l C_{r-s}^{k-l} / C_r^k.$$

A. Addition and multiplication of probabilities. Conditional probability

Definition: A **union** of two events $A \cup B$ is an event C such that **at least one** of the two events takes place. We denote the probability of C as $P(C) \equiv P(A \cup B)$.
Example: It is going to be warm tomorrow sometime during the day. Let us call it an event C. It is a sum of events "A=it is going to be warm tomorrow morning" and "B=it will be warm tomorrow afternoon".

If the events A and B are *incompatible*, we have the rule

$$\boxed{P(A \cup B) = P(A) + P(B)}. \tag{5}$$

Definition: An **intersection** of two events A and B is an event C such that **both** A **and** B take place. We denote the probability of C as $P(C) \equiv P(A \cap B)$. The product indicates a two-event overlap (the events may occur simultaneously).
Example: It is going to be warm tomorrow throughout the day. Let us call it an event C. It is a product of events "A= it is going to be warm tomorrow morning" and "B=it will be warm tomorrow afternoon".

If the events A and B may overlap, the rule of Eq. (5) is modified to

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}. \tag{6}$$

Definition: Two events A and $\overline{A}$ are **complementary** if only two mutually exclusive alternatives are possible: the occurrence of A and the occurrence of not A.
Example: The appearance of tails in a coin flip experiment is complementary to the appearance of heads.
Definition: Two or more events are mutually exclusive if the occurrence of one implies non-occurrence of the others. Note that complementarity and mutual exclusiveness are not identical: two events can be mutually exclusive, but not complementary.
Example: The appearance of 2 and 6 when a dice is rolled are mutually exclusive, but not

complementary (1, 3,4, and 5 can also show up).

It follows from the definition that

$$\boxed{P(\overline{A}) + P(A) = 1 \implies P(\overline{A}) = 1 - P(A)}. \tag{7}$$

Hereafter we will refer to the intersection of two events as a "product" and denote $A \cap B$ as simply AB.

Definition: The **conditional** probability of A **given** B is defined as

$$\boxed{P(A/B) \equiv \frac{P(AB)}{P(B)}} \tag{8}$$

It follows at once that the probability of a product of two events can be expressed in terms of the conditional probability as

$$\boxed{P(AB) = P(A/B)P(B) = P(B/A)P(A)}. \tag{9}$$

Definition: The two events A and B are **independent** if the occurrence of A does **not** affect the occurrence of B.

Example: A=heads show up on the first coin toss and B=tails show up on the second toss are independent events.

It follows from the conditional probability definition that

$$P(A/B) = P(A); \quad P(B/A) = P(B), \iff \text{independent A and B.} \tag{10}$$

It follows at once from Eqs. (9) and (10) that

$$\boxed{P(AB) = P(A)P(B), \iff \text{independent A and B}}. \tag{11}$$

B. Practice Problems on Sum and Product of Probabilities

Problem 1. Urn and balls revisited: *An urn contains a white and b black balls. Someone randomly draws two balls from the urn. What is the probability that the balls are white?*

Solution–Let us introduce the events

- H_1 = the first ball is white;
- H_2 = the second ball is white.

It follows by definition that $P(H_1) = a/(a+b)$. By the same token, we can determine the conditional probability that the second ball is white, given the first one is, $P(H_2/H_1) = (a-1)/(a+b-1)$. It then follows from Eq. (9) that

$$P(H_1 H_2) = P(H_1)P(H_2/H_1) = \frac{a(a-1)}{(a+b)(a+b-1)}. \tag{12}$$

Compare with Example 2!

Problem 2. Urn and balls yet again: *An urn contains a white and b black balls. Someone randomly draws two balls from the urn. What is the probability that the balls will have different colours?*

Solution– Method 1. Introducing the events

- H_1 = a white ball is drawn;
- H_2 = a black ball is drawn.

The sought outcome is obtained if *either* one draws first a black ball, followed by a white ball, *or* the other way around. The probability of this is

$$P_s = P(H_1, H_2) + P(H_2, H_1).$$

However,

$$P(H_1, H_2) = P(H_1)P(H_2/H_1) = \frac{a}{a+b}\frac{b}{(a+b-1)} = \frac{ab}{(a+b)(a+b-1)},$$

By the same token,

$$P(H_2, H_1) = P(H_2)P(H_1/H_2) = \frac{b}{a+b}\frac{a}{(a+b-1)} = \frac{ab}{(a+b)(a+b-1)},$$

Thus,

$$P_s = \frac{2ab}{(a+b)(a+b-1)}.$$

Method 2. The problem boils down to determining the number of combinations. The number of favourite outcomes corresponds to the number of combinations of a white and black balls out of a white and b black which is $S = ab$. The total number of outcomes is a number of pairs out of $a+b$ balls in no particular order, $N = C_2^{a+b}$. Hence,

$$P_s = S/N = ab/C_2^{a+b} = \frac{2!ab(a+b-2)!}{(a+b)!} = \frac{2ab}{(a+b)(a+b-1)}.$$

Problem 3. Urn and balls with a twist: *An urn contains a white and b black balls. Someone randomly draws all balls from the urn one by one. What is the probability that the second ball coming out of the urn is white?*

Solution–Introducing the events

- H_1 = a white ball is drawn first;
- H_2 = a black ball is drawn first;
- H_3 = a white ball is drawn second.

It follows from Eqs. (5) and (9) that

$$P(H_3) = P(H_1)P(H_3/H_1) + P(H_2)P(H_3/H_2).$$

We know that

$$P(H_1) = \frac{a}{a+b} \qquad P(H_2) = \frac{b}{a+b};$$

and

$$P(H_3/H_1) = \frac{a-1}{a+b-1}, \qquad P(H_3/H_2) = \frac{a}{a+b-1}.$$

It follows that

$$P(H_3) = \frac{a(a-1)}{(a+b)(a+b-1)} + \frac{ab}{(a+b)(a+b-1)} = \frac{a(a+b-1)}{(a+b)(a+b-1)} = \frac{a}{a+b}.$$

Problem 4. Urn and three kinds of balls: *An urn contains a white, b black and c red balls. Someone randomly draws three balls. What is the probability that at least two will be of the same colour?*

Solution–Let us introduce the event A = at least two randomly picked balls are of the same colour and find out the probability of a complementary event $\overline{A}$ = none of the balls are of the same colour. That's the easiest because we just have to work out the number of combinations of three balls of different colours which is abc. Thus,

$$P(\overline{A}) = \frac{abc}{C_3^{a+b+c}} = \frac{3!abc(a+b+c-3)!}{(a+b+c)!} = \frac{6abc}{(a+b+c)(a+b+c-1)(a+b+c-2)},$$

and, finally,

$$P(A) = 1 - \frac{6abc}{(a+b+c)(a+b+c-1)(a+b+c-2)}.$$

This is much easier than trying to work out various conditional probabilities of three events!

Problem 5. Target detection by multiple radar stations: *There are m radar stations each of which can independently detect an object with the probability p over one cycle. Object detection at each cycle is independent of the other cycles. Each station makes n cycles over the time period T. Determine the probability that during the time T*

a) the object will be detected by at least one station;

b) the object will be detected by all stations.

<u>Solution</u>–a). Introduce events

- A=object is detected by at least one station;
- $\overline{A}$=object is not detected;
- H=object is detected by one station over one cycle;
- $\overline{H}$=object is not detected by a station over a cycle;
- B=object is not detected by a station.

Clearly, $P(H) = p$ and $P(\overline{H}) = 1 - p$. Since all cycles are independent, $P(B) = (1 - p)^n$. Further, all stations are independent, implying that

$$P(\overline{A}) = [P(B)]^m = (1 - p)^{mn} \implies P(A) = 1 - (1 - p)^{mn}.$$

b) Introducing a different set of events,

- A=object is detected by all stations;
- H_1=object is detected by a station over at least a cycle;
- $\overline{H}_1$=object is not detected by a station.

As all cycles are independent, $P(\overline{H}_1) = (1 - p)^n$, implying

$$P(H) = 1 - P(\overline{H}_1) = 1 - (1 - p)^n.$$

Since there are m independent stations and each has to detect the object,

$$P(A) = [P(H)]^m = [1 - (1 - p)^n]^m.$$

Problem 6. Multiple target detection by multiple radar stations: *There are k space objects in a group. Each can be detected by a radar station with the probability p*

independently of the others. There are m independent radar stations to search for the group. What is the probability that not all objects in the group will be detected?

Solution–The event A=not all objects in the group are detected is equivalent to A=at least one object remains undetected. Introducing the events then

- A=at least one object remains undetected;
- $\overline{A}$=all objects are detected;
- H=object is not detected by any station;
- $\overline{H}$=object is detected by at least one station.

Since there are m independent stations,

$$P(H) = (1-p)^m \implies P(\overline{H}) = 1-(1-p)^m.$$

Next, since there are k objects to be independently detected, A occurs if any object is detected by at least one station, i. e.,

$$P(A) = [P(\overline{H})]^k = [1-(1-p)^m]^k,$$

Finally,

$$P(\overline{A}) = 1 - P(A) = 1 - [1-(1-p)^m]^k.$$

C. Bayes's rule

Suppose an event A can occur if at least one event of the set $\{H_i\}$, which we call hypothesis, can be realized. The *apriori* probability of A is then

$$P(A) = \sum_i P(H_i)P(A/H_i). \tag{13}$$

Suppose further we are interested in figuring out an *aposteriori* probability of a hypothesis H_i realization, given the event A occurs. The probability of both event A and hypothesis H_i to happen can be expressed as

$$P(AH_i) = P(H_i)P(A/H_i) = P(A)P(H_i/A). \tag{14}$$

The Bayes rule then follows

$$\boxed{P(H_i/A) = \frac{P(H_i)}{P(A)}P(A/H_i) = \frac{P(H_i)P(A/H_i)}{\sum_i P(H_i)P(A/H_i)}}. \tag{15}$$

Problem 1. Urns and balls with a twist: *There are three identical urns. There are a white balls and b black ones in the first urn. There are c white balls and d black balls in the second and the third contains only white balls. Someone draws a ball at random from a randomly selected urn.*

a) What is the probability that the ball is white?

b) Determine the probability that the ball comes from the third urn, given it is white.

<u>Solution</u>–a) Introducing our hypotheses:

- H_j=urn j is picked, $j = 1, 2, 3$
- W=white ball is picked.

By the sum and product rules,

$$P(W) = \sum_{j=1}^{3} P(H_j)P(W/H_j).$$

Our base probabilities are

$$P(H_j) = 1/3,$$

and

$$P(W/H_1) = \frac{a}{a+b}, \quad P(W/H_2) = \frac{c}{c+d}, \quad P(W/H_3) = 1.$$

It then follows that

$$P(W) = \frac{1}{3}\left(1 + \frac{a}{a+b} + \frac{c}{c+d}\right).$$

b) Using Bayes' rule,

$$P(H_3/W) = \frac{P(H_3)}{P(W)}P(W/H_3) = \frac{1}{1 + \frac{a}{a+b} + \frac{c}{c+d}}.$$

Problem 2. Bayesian quality control: *A device consists of two parts. The device is operational only if both parts are. The probability that each part remains operational over the time period t is p_j, $j = 1, 2$. The device is tested over the time t and found to fail. Determine the probability that the first part fails while the second remains operational.*

<u>Solution</u>–Introducing our hypotheses:

- H_j=part j, ($j = 1, 2$), fails;
- A=device fails.

We are interested in the probability $P(H_1\overline{H}_2/A)$. It can be determined using Bayes' rule,

$$P(H_1\overline{H}_2/A) = \frac{P(H_1\overline{H}_2)}{P(A)}P(A/H_1\overline{H_2})$$

Assuming each part works independently,

$$P(H_1\overline{H}_2) = p_2(1 - p_1).$$

Since both parts are required for the device to remain operational, $P(A/H_1\overline{H_2}) = 1$. Let us now work out the probability the device remains operational, $P(\overline{A})$. This happens only if both parts remain operational, i.e.,

$$P(\overline{A}) = p_1p_2, \implies P(A) = 1 - P(\overline{A}) = 1 - p_1p_2.$$

Putting it all together,

$$P(H_1\overline{H}_2/A) = \frac{p_2(1 - p_1)}{1 - p_1p_2}.$$

D. Repeated Trials and Bernoulli formula

Consider a series of events such that each event can have only two outcomes we will refer to as "success" with the probability p_s and "failure" with the probability of $p_f = 1 - p_s$. Such trials are called **binary**.

Example: Flipping a fair coin: heads/tails can be treated as success/failure depending on the problem formulation, $p_s = p_f = 1/2$.

The Bernoulli formula pertains to a probability of getting m successes in n binary trials. It is very straightforward to derive. Suppose we mark the outcome of each trial as a sequence of arrows pointing up and down. Each up-arrow corresponds to a success and down-arrow to a failure. The probability to get a particular string is simply $p_s^m p_f^{n-m}$. However, all such strings have to be counted and their number corresponds to C_m^n because the arrow order is irrelevant. Thus,

$$\boxed{P(m, n) = C_m^n p_s^m p_f^{n-m} = C_m^n p_s^m (1 - p_s)^{n-m}}. \tag{16}$$

The key point in applying the Bernoulli formula is to identify the probability of success of an individual trial.

Problem 1. Quality control a-la Bernoulli: *A production line generates an output. The output can contain defective items. A defective item can be produced with a probability* r*. The quality check can detect a defect with the probability* p*. A party of* n *items is chosen for quality check. Determine the probability that*

a) none of the items in the party contains defects;

b) exactly two items out of n *come out to be defective;*

c) at least two items contain defects.

Solution–The problem corresponds to a Bernoulli trial with the probability of success (defective item found) $p_s = rp$ and failure (an item is good), $p_f = 1 - rp$. Hence,

a) $P(n, n) = (1 - rp)^n$; b) $P(2, n) = C_2^n r^2 p^2 (1 - rp)^{n-2}$; c) Let us introduce two events,

- C=at least two items are defective;
- $\overline{C}$=none or one item is defective.

It is much easier to work out $P(\overline{C})$:

$$P(\overline{C}) = (1 - rp)^n + C_1^n rp(1 - rp)^{n-1} = (1 - rp)^n + nrp(1 - rp)^{n-1}.$$

Finally,

$$P(C) = 1 - P(\overline{C}), \implies P(C) = 1 - (1 - rp)^n + nrp(1 - rp)^{n-1}.$$

Problem 2. Air defence against bombers: N *bombers take part in an air raid. The air defences allocate 2 fighters to shoot down each bomber. Each fighter can take down a bomber with the probability* p*, independently of the other fighters. Find the probability that*

a) A=exactly three bombers are shot down;

b) B=at least two bombers are downed.

Solution–This is a Bernoulli trial. A bomber returns unscathed if both fighters assigned to it miss; the probability of that is simply $p_f = (1 - p)^2$. Otherwise, the bomber is downed. Thus,

$$p_s = 1 - p_f = 1 - (1 - p)^2.$$

a) We require 3 successes in N trials,

$$P(A) = P(3, N) = C_3^N [1 - (1 - p)^2]^3 (1 - p)^{2(N-3)}.$$

b) It is easier to figure out the probability of a complementary event, $\overline{B}$=none or one bomber is shot down,

$$P(\overline{B}) = P(0, N) + P(1, N) = (1-p)^{2N} + N[1-(1-p)^2](1-p)^{2(N-1)}.$$

Finally,

$$P(B) = 1 - P(\overline{B}) = 1 - (1-p)^{2(N-1)}\{(1-p)^2 + N[1-(1-p)^2]\}.$$