Math bootcamp: Complex numbers & elementary functions of complex variables

Sergey A. Ponomarenko

Department of Electrical & Computer Engineering, Dalhousie University

I. COMPLEX NUMBERS AND PHASORS

Definition. A **complex number** z can be expressed in the so-called rectangular form as

$$\boxed{z = u + jv},$$

where $j = \sqrt{-1}$ and u and v are real numbers. Alternatively, it can be expressed in the polar form as

$$\boxed{z = re^{j\phi} = r(\cos\phi + j\sin\phi)},$$

where the magnitude r and phase ϕ can be written as

$$\boxed{r = \sqrt{u^2 + v^2}, \qquad \phi = \tan^{-1} v/u}.$$

Geometrically, z can be represented as a ray in the uv plane making the angle ϕ with the u-axis, see Fig. 1.8.

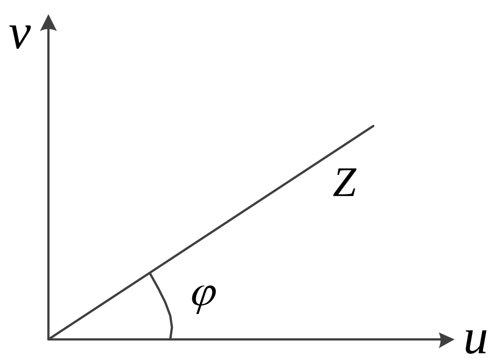

FIG. 1. Polar form of a complex number.

Given two complex numbers, $z_1 = u_1 + jv_1 = r_1 e^{j\phi_1}$ and $z_2 = u_2 + jv_2 = r_2 e^{j\phi_2}$, the result of their addition or subtraction can be most easily expressed in the rectangular form:

$$z_1 \pm z_2 = u_1 \pm u_2 + j(v_1 \pm v_2).$$

On the other hand, their multiplication and division are more naturally expressed in the polar form as

$$z_1 z_2 = r_1 r_2 e^{j(\phi_1+\phi_2)}, \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{j(\phi_1-\phi_2)}.$$

One can also introduce complex conjugation by the definition

$$\boxed{z^* = u - jv = re^{-j\phi}}.$$

In the polar form, a complex number is not uniquely defined such that

$$z = re^{j\phi}e^{j2\pi k}, \qquad k = 0, \pm 1, \pm 2, \pm 3 \ldots.$$

This is because $e^{j2\pi k} = 1$ for any integer k. The latter form comes in handy whenever we want to find roots of a complex number. In general all nth roots can be represented as

$$z^{1/n} = r^{1/n}e^{j\phi/n}e^{j2\pi k/n}.$$

For example, if $n = 2$, there are only two distinct roots corresponding to $k = 0$ and $k = 1$; since $e^{j0} = 1$ and $e^{j\pi} = \cos\pi + j\sin\pi = -1$, we obtain

$$\sqrt{z} = \pm\sqrt{r}e^{j\phi/2}.$$

Example 1. The complex impedance of a monochromatic electromagnetic wave of frequency ω, propagating in a lossy medium is defined as

$$\eta = \sqrt{\frac{\mu/\epsilon}{1 + \frac{j\sigma}{\epsilon\omega}}}.$$

Here μ, η and σ are constitutive parameters of the medium. Express η in the polar form.
Solution. Multiplying the numerator and denominator inside the square root by $(1 - j\sigma/\epsilon\omega)$, *we obtain*

$$\eta = \frac{\sqrt{\mu/\epsilon}\,\left(1 - \frac{j\sigma}{\epsilon\omega}\right)^{1/2}}{\left[1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/2}} = \frac{\sqrt{\mu/\epsilon}\,e^{j\theta_\eta}}{\left[1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/4}} = |\eta|e^{j\theta_\eta},$$

where

$$|\eta| = \frac{\sqrt{\mu/\epsilon}}{\left[1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2\right]^{1/4}}, \qquad \tan 2\theta_\eta = \frac{\sigma}{\epsilon\omega}.$$

Definition. A **time-harmonic signal** varies sinusoidally with time.
Definition. A **phasor** represents a complex signal with a time-harmonic phase.
Thus any physical time-harmonic signal $\psi(t) = a\cos(\omega t + \theta)$, where ω and θ are constant frequency and initial phase, respectively, can be represented in terms of a complex phasor $\psi_0 e^{j\omega t}$ as

$$\boxed{\psi(t) = \mathrm{Re}(\psi_0 e^{-j\omega t})}.$$

Here Re denotes the real part of the complex signal and the complex amplitude ψ_0 can be represented as

$$\boxed{\psi_0 = a\, e^{j\theta}},$$

where a is a real amplitude.

II. ELEMENTARY FUNCTIONS OF COMPLEX VARIABLES

Consider a regular **real** function of a complex variable z. How can one determine its complex conjugate, $f^*(z)$? A regular function can be expanded into a Taylor series as

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{z^n}{n!},$$

where $f^{(n)}(0)$ stands for the nth derivative of f at $z = 0$. Obviously,

$$(z^n)^* = r^r (e^{jn\phi})^* = r^n e^{-jn\phi} = z^{*n},$$

implying that

$$f^*(z) = f(z^*). \tag{1}$$

Let us up the ante by considering a **complex** function $F(z)$. What's its complex conjugate? One can rewrite F in the rectangular form,

$$F(z) = U(z) + jV(z), \implies [F(z)]^* = [U(z)]^* - j[V(z)]^*.$$

Since both U and V are real functions, the rule of Eq. (1) applies to each of them, implying the following rule for $F(z)$,

$$\boxed{[F(z)]^* = F^*(z^*)}. \tag{2}$$

Definition. The exponential function of a complex variable z is defined as

$$e^z = e^{u+jv} = e^u e^{jv} = e^u(\cos v + j\sin v).$$

Definition. Trigonometric functions of a complex variable are defined as

$$\cos z \equiv \frac{e^{jz} + e^{-jz}}{2} \tag{3}$$

$$\sin z \equiv \frac{e^{jz} - e^{-jz}}{2j}, \tag{4}$$

and

$$\tan z \equiv \frac{\sin z}{\cos z}.$$

It can be read off from the definition (3) and (4) by inspection that

$$\sin^2 z + \cos^2 z = 1.$$

Example 2. Show that $|\cos z|^2 = \frac{1}{2}(cos2u + \cosh 2v)$.

Solution. $|\cos z|^2 = [\cos z]^* \cos z$. *Using Eq. (2), we obtain*

$$|\cos z|^2 = \frac{1}{4}(e^{jz^*} + e^{-jz^*})(e^{jz} + e^{-jz}) = \frac{1}{4}[e^{j(z+z^*)} + e^{-j(z+z^*)} + e^{j(z-z^*)} + e^{-j(z-z^*)}].$$

Using the fact that

$$z + z^* = 2u, \qquad z - z^* = 2jv,$$

and the definition of the cosine, Eq. (3), we arrive at the answer.

Exercise 1. Show that $|\sin z|^2 = \frac{1}{2}(\cosh 2v - \cos 2u)$.

Definition. Hyperbolic functions of a complex variable z are defined as

$$\cosh z \equiv \frac{e^z + e^{-z}}{2} \qquad \sinh z \equiv \frac{e^z - e^{-z}}{2},$$

and

$$\tanh z \equiv \frac{\sinh z}{\cosh z}.$$

Notice that

$$\cosh z = \cos iz, \quad \sinh z = -j \sin jz, \quad \tanh z = -j \tan jz.$$

Hence all properties of the hyperbolic functions follow from those of the trigonometric ones. For example,

$$\cosh^2 z - \sinh^2 z = 1.$$

Definition. We define a logarithm of z, such that

$$w = \log z,$$

such that

$$e^w = z.$$

Consider $w = u + jv$ and $z = re^{j\phi}$, we arrive at

$$e^w = e^u e^{jv} = re^{j\phi}, \implies r = e^u, \qquad v = \phi + 2\pi k,$$

where $k = 0, \pm 1, \pm 2, \ldots$. Hence,

$$w = \log z = \log r + j\phi + 2\pi k j = \log|z| + j\arg(z) + 2\pi k j.$$

Definition. We define inverse hyperbolic functions as follows

$$w = \sinh^{-1} z, \quad \Longleftrightarrow z = \sinh w,$$

and

$$w = \cosh^{-1} z, \quad \Longleftrightarrow z = \cosh w.$$

Example 3. Show that $\sinh^{-1} z = \log(z + \sqrt{z^2+1})$ and $\cosh^{-1} z = \log(z + \sqrt{z^2-1})$.

Solution.

$$z = \sinh w = \tfrac{1}{2}(e^w - e^{-w}); \qquad z = \cosh w = \tfrac{1}{2}(e^w + e^{-w}).$$

On multiplying both sides of the equation with $2e^w$, it follows that

$$s^2 \pm 1 = 2zs \quad \Longrightarrow \quad s^2 - 2zs \pm 1 = 0,$$

where we introduced $s = e^w$; +(-) sign corresponds to cosh(sinh). *Solving the quadratic equation, we obtain two roots,*

$$s_1 = z + \sqrt{z^2 \mp 1}, \qquad s_2 = z - \sqrt{z^2 \mp 1}.$$

Consider the purely real case when $v = 0$ and take the limit $u \longrightarrow \pm\infty$. It follows from the definition of real valued hyperbolic functions that

$$u \longrightarrow \pm\infty \quad \Longrightarrow \sinh u(\cosh u) \longrightarrow \infty,$$

We see that only the first root respects this limit. In other words,

$$s = e^w = z + \sqrt{z^2 \mp 1},$$

Thus,

$$w = \cosh^{-1} z = \log(z + \sqrt{z^2-1}); \qquad w = \sinh^{-1} z = \log(z + \sqrt{z^2+1}).$$